

## 4.1 Antiderivatives and Indefinite Integration

- Write the general solution of a differential equation and use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular solution of a differential equation.

### Antiderivatives

To find a function  $F$  whose derivative is  $f(x) = 3x^2$ , you might use your knowledge of derivatives to conclude that

$$F(x) = x^3 \quad \text{because} \quad \frac{d}{dx}[x^3] = 3x^2.$$

The function  $F$  is an *antiderivative* of  $f$ .

#### Definition of Antiderivative

A function  $F$  is an **antiderivative** of  $f$  on an interval  $I$  when  $F'(x) = f(x)$  for all  $x$  in  $I$ .

Note that  $F$  is called *an* antiderivative of  $f$ , rather than *the* antiderivative of  $f$ . To see why, observe that

$$F_1(x) = x^3, \quad F_2(x) = x^3 - 5, \quad \text{and} \quad F_3(x) = x^3 + 97$$

are all antiderivatives of  $f(x) = 3x^2$ . In fact, for any constant  $C$ , the function  $F(x) = x^3 + C$  is an antiderivative of  $f$ .

#### THEOREM 4.1 Representation of Antiderivatives

If  $F$  is an antiderivative of  $f$  on an interval  $I$ , then  $G$  is an antiderivative of  $f$  on the interval  $I$  if and only if  $G$  is of the form  $G(x) = F(x) + C$ , for all  $x$  in  $I$  where  $C$  is a constant.

**Proof** The proof of Theorem 4.1 in one direction is straightforward. That is, if  $G(x) = F(x) + C$ ,  $F'(x) = f(x)$ , and  $C$  is a constant, then

$$G'(x) = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x).$$

To prove this theorem in the other direction, assume that  $G$  is an antiderivative of  $f$ . Define a function  $H$  such that

$$H(x) = G(x) - F(x).$$

For any two points  $a$  and  $b$  ( $a < b$ ) in the interval,  $H$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . By the Mean Value Theorem,

$$H'(c) = \frac{H(b) - H(a)}{b - a}$$

for some  $c$  in  $(a, b)$ . However,  $H'(c) = 0$ , so  $H(a) = H(b)$ . Because  $a$  and  $b$  are arbitrary points in the interval, you know that  $H$  is a constant function  $C$ . So,  $G(x) - F(x) = C$  and it follows that  $G(x) = F(x) + C$ .

See [LarsonCalculus.com](http://LarsonCalculus.com) for Bruce Edwards's video of this proof.

#### Exploration

##### Finding Antiderivatives

For each derivative, describe the original function  $F$ .

a.  $F'(x) = 2x$

b.  $F'(x) = x$

c.  $F'(x) = x^2$

d.  $F'(x) = \frac{1}{x^2}$

e.  $F'(x) = \frac{1}{x^3}$

f.  $F'(x) = \cos x$

What strategy did you use to find  $F$ ?

Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that

$$D_x[x^2] = 2x$$

you can represent the family of *all* antiderivatives of  $f(x) = 2x$  by

$$G(x) = x^2 + C \quad \text{Family of all antiderivatives of } f(x) = 2x$$

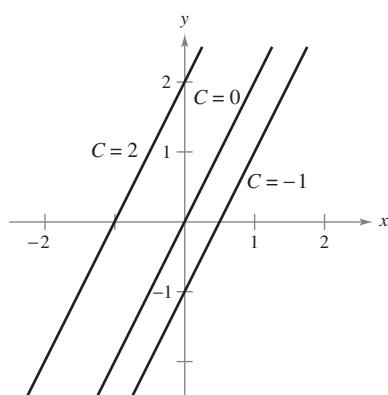
where  $C$  is a constant. The constant  $C$  is called the **constant of integration**. The family of functions represented by  $G$  is the **general antiderivative** of  $f$ , and  $G(x) = x^2 + C$  is the **general solution** of the differential equation

$$G'(x) = 2x. \quad \text{Differential equation}$$

A **differential equation** in  $x$  and  $y$  is an equation that involves  $x$ ,  $y$ , and derivatives of  $y$ . For instance,

$$y' = 3x \quad \text{and} \quad y' = x^2 + 1$$

are examples of differential equations.



Functions of the form  $y = 2x + C$

**Figure 4.1**

### EXAMPLE 1 Solving a Differential Equation

Find the general solution of the differential equation  $y' = 2$ .

**Solution** To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x. \quad \text{2x is an antiderivative of 2.}$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C. \quad \text{General solution}$$

The graphs of several functions of the form  $y = 2x + C$  are shown in Figure 4.1. ■

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x) dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign  $\int$ . The general solution is denoted by

••••• **REMARK** In this text, the notation  $\int f(x) dx = F(x) + C$  means that  $F$  is an antiderivative of  $f$  on an interval.

$$y = \int f(x) dx = F(x) + C.$$

Diagram illustrating the components of the integral equation:

- Variable of integration** points to  $dx$ .
- Constant of integration** points to  $C$ .
- Integrand** points to  $f(x)$ .
- An antiderivative of  $f(x)$**  points to  $F(x)$ .

The expression  $\int f(x) dx$  is read as the *antiderivative of  $f$  with respect to  $x$* . So, the differential  $dx$  serves to identify  $x$  as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

## Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting  $F'(x)$  for  $f(x)$  in the indefinite integration definition to obtain

$$\int F'(x) dx = F(x) + C.$$

Integration is the “inverse” of differentiation.

Moreover, if  $\int f(x) dx = F(x) + C$ , then

$$\frac{d}{dx} \left[ \int f(x) dx \right] = f(x).$$

Differentiation is the “inverse” of integration.

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

### Basic Integration Rules

#### Differentiation Formula

$$\frac{d}{dx} [C] = 0$$

$$\frac{d}{dx} [kx] = k$$

$$\frac{d}{dx} [kf(x)] = kf'(x)$$

$$\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\frac{d}{dx} [x^n] = nx^{n-1}$$

$$\frac{d}{dx} [\sin x] = \cos x$$

$$\frac{d}{dx} [\cos x] = -\sin x$$

$$\frac{d}{dx} [\tan x] = \sec^2 x$$

$$\frac{d}{dx} [\sec x] = \sec x \tan x$$

$$\frac{d}{dx} [\cot x] = -\csc^2 x$$

$$\frac{d}{dx} [\csc x] = -\csc x \cot x$$

#### Integration Formula

$$\int 0 dx = C$$

$$\int k dx = kx + C$$

$$\int kf(x) dx = k \int f(x) dx$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \quad \text{Power Rule}$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

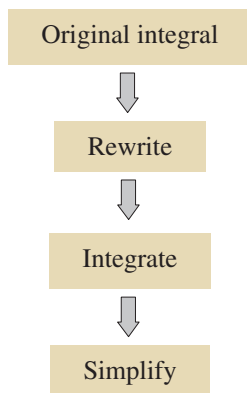
$$\int \csc x \cot x dx = -\csc x + C$$

Note that the Power Rule for Integration has the restriction that  $n \neq -1$ . The evaluation of

$$\int \frac{1}{x} dx$$

must wait until the introduction of the natural logarithmic function in Chapter 5.

••••• ▶ **REMARK** In Example 2, note that the general pattern of integration is similar to that of differentiation.

**EXAMPLE 2****Describing Antiderivatives**

$$\begin{aligned}
 \int 3x \, dx &= 3 \int x \, dx && \text{Constant Multiple Rule} \\
 &= 3 \int x^1 \, dx && \text{Rewrite } x \text{ as } x^1. \\
 &= 3 \left( \frac{x^2}{2} \right) + C && \text{Power Rule } (n = 1) \\
 &= \frac{3}{2} x^2 + C && \text{Simplify.}
 \end{aligned}$$

The antiderivatives of  $3x$  are of the form  $\frac{3}{2}x^2 + C$ , where  $C$  is any constant. ■

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration. For instance, in Example 2, the solution could have been written as

$$\int 3x \, dx = 3 \int x \, dx = 3 \left( \frac{x^2}{2} + C \right) = \frac{3}{2} x^2 + 3C.$$

Because  $C$  represents *any* constant, it is both cumbersome and unnecessary to write  $3C$  as the constant of integration. So,  $\frac{3}{2}x^2 + 3C$  is written in the simpler form  $\frac{3}{2}x^2 + C$ .

▶ **TECHNOLOGY** Some software programs, such as *Maple* and *Mathematica*, are capable of performing integration symbolically. If you have access to such a symbolic integration utility, try using it to evaluate the indefinite integrals in Example 3.

**EXAMPLE 3****Rewriting Before Integrating**

••••▶ See [LarsonCalculus.com](http://LarsonCalculus.com) for an interactive version of this type of example.

Original Integral	Rewrite	Integrate	Simplify
a. $\int \frac{1}{x^3} \, dx$	$\int x^{-3} \, dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b. $\int \sqrt{x} \, dx$	$\int x^{1/2} \, dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3} x^{3/2} + C$
c. $\int 2 \sin x \, dx$	$2 \int \sin x \, dx$	$2(-\cos x) + C$	$-2 \cos x + C$

••••• ▶ **REMARK** The basic integration rules allow you to integrate any polynomial function.

**EXAMPLE 4****Integrating Polynomial Functions**

$$\begin{aligned}
 \text{a. } \int dx &= \int 1 \, dx && \text{Integrand is understood to be 1.} \\
 &= x + C && \text{Integrate.} \\
 \text{b. } \int (x + 2) \, dx &= \int x \, dx + \int 2 \, dx \\
 &= \frac{x^2}{2} + C_1 + 2x + C_2 && \text{Integrate.} \\
 &= \frac{x^2}{2} + 2x + C && C = C_1 + C_2
 \end{aligned}$$

The second line in the solution is usually omitted.

$$\begin{aligned}
 \text{c. } \int (3x^4 - 5x^2 + x) \, dx &= 3 \left( \frac{x^5}{5} \right) - 5 \left( \frac{x^3}{3} \right) + \frac{x^2}{2} + C \\
 &= \frac{3}{5} x^5 - \frac{5}{3} x^3 + \frac{1}{2} x^2 + C
 \end{aligned}$$
■



**REMARK** Before you begin the exercise set, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits one of the basic integration rules.

### EXAMPLE 5 Rewriting Before Integrating

$$\begin{aligned}
 \int \frac{x+1}{\sqrt{x}} dx &= \int \left( \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right) dx && \text{Rewrite as two fractions.} \\
 &= \int (x^{1/2} + x^{-1/2}) dx && \text{Rewrite with fractional exponents.} \\
 &= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C && \text{Integrate.} \\
 &= \frac{2}{3}x^{3/2} + 2x^{1/2} + C && \text{Simplify.} \\
 &= \frac{2}{3}\sqrt{x}(x+3) + C
 \end{aligned}$$

When integrating quotients, do not integrate the numerator and denominator separately. This is no more valid in integration than it is in differentiation. For instance, in Example 5, be sure you understand that

$$\int \frac{x+1}{\sqrt{x}} dx = \frac{2}{3}\sqrt{x}(x+3) + C$$

is not the same as

$$\frac{\int (x+1) dx}{\int \sqrt{x} dx} = \frac{\frac{1}{2}x^2 + x + C_1}{\frac{2}{3}x\sqrt{x} + C_2}.$$

### EXAMPLE 6 Rewriting Before Integrating

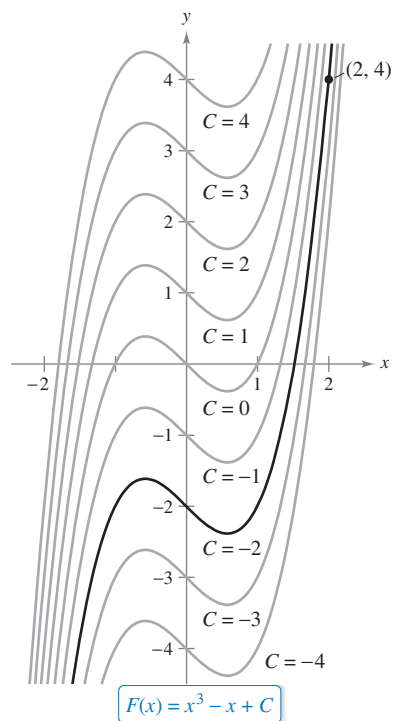
$$\begin{aligned}
 \int \frac{\sin x}{\cos^2 x} dx &= \int \left( \frac{1}{\cos x} \right) \left( \frac{\sin x}{\cos x} \right) dx && \text{Rewrite as a product.} \\
 &= \int \sec x \tan x dx && \text{Rewrite using trigonometric identities.} \\
 &= \sec x + C && \text{Integrate.}
 \end{aligned}$$

### EXAMPLE 7 Rewriting Before Integrating

Original Integral	Rewrite	Integrate	Simplify
a. $\int \frac{2}{\sqrt{x}} dx$	$2 \int x^{-1/2} dx$	$2 \left( \frac{x^{1/2}}{1/2} \right) + C$	$4x^{1/2} + C$
b. $\int (t^2 + 1)^2 dt$	$\int (t^4 + 2t^2 + 1) dt$	$\frac{t^5}{5} + 2 \left( \frac{t^3}{3} \right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
c. $\int \frac{x^3 + 3}{x^2} dx$	$\int (x + 3x^{-2}) dx$	$\frac{x^2}{2} + 3 \left( \frac{x^{-1}}{-1} \right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
d. $\int \sqrt[3]{x}(x-4) dx$	$\int (x^{4/3} - 4x^{1/3}) dx$	$\frac{x^{7/3}}{7/3} - 4 \left( \frac{x^{4/3}}{4/3} \right) + C$	$\frac{3}{7}x^{7/3} - 3x^{4/3} + C$

As you do the exercises, note that you can check your answer to an antidifferentiation problem by differentiating. For instance, in Example 7(a), you can check that  $4x^{1/2} + C$  is the correct antiderivative by differentiating the answer to obtain

$$D_x[4x^{1/2} + C] = 4 \left( \frac{1}{2} \right) x^{-1/2} = \frac{2}{\sqrt{x}}. \quad \text{Use differentiation to check antiderivative.}$$



The particular solution that satisfies the initial condition  $F(2) = 4$  is  $F(x) = x^3 - x - 2$ .

Figure 4.2

## Initial Conditions and Particular Solutions

You have already seen that the equation  $y = \int f(x) dx$  has many solutions (each differing from the others by a constant). This means that the graphs of any two antiderivatives of  $f$  are vertical translations of each other. For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1) dx = x^3 - x + C \quad \text{General solution}$$

for various integer values of  $C$ . Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$

In many applications of integration, you are given enough information to determine a **particular solution**. To do this, you need only know the value of  $y = F(x)$  for one value of  $x$ . This information is called an **initial condition**. For example, in Figure 4.2, only one curve passes through the point  $(2, 4)$ . To find this curve, you can use the general solution

$$F(x) = x^3 - x + C \quad \text{General solution}$$

and the initial condition

$$F(2) = 4. \quad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that

$$F(2) = 8 - 2 + C = 4$$

which implies that  $C = -2$ . So, you obtain

$$F(x) = x^3 - x - 2. \quad \text{Particular solution}$$

### EXAMPLE 8 Finding a Particular Solution

Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition  $F(1) = 0$ .

**Solution** To find the general solution, integrate to obtain

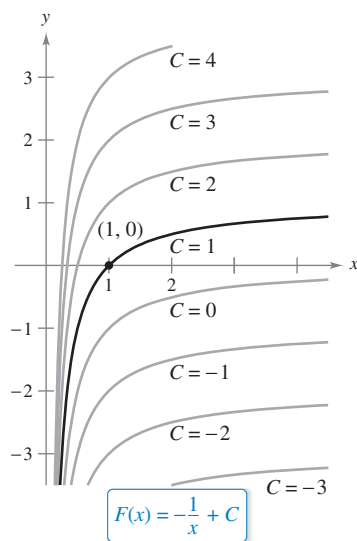
$$\begin{aligned} F(x) &= \int \frac{1}{x^2} dx && F(x) = \int F'(x) dx \\ &= \int x^{-2} dx && \text{Rewrite as a power.} \\ &= \frac{x^{-1}}{-1} + C && \text{Integrate.} \\ &= -\frac{1}{x} + C, \quad x > 0. && \text{General solution} \end{aligned}$$

Using the initial condition  $F(1) = 0$ , you can solve for  $C$  as follows.

$$F(1) = -\frac{1}{1} + C = 0 \Rightarrow C = 1$$

So, the particular solution, as shown in Figure 4.3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \quad \text{Particular solution}$$



The particular solution that satisfies the initial condition  $F(1) = 0$  is  $F(x) = -(1/x) + 1, x > 0$ .

Figure 4.3

So far in this section, you have been using  $x$  as the variable of integration. In applications, it is often convenient to use a different variable. For instance, in the next example, involving *time*, the variable of integration is  $t$ .

### EXAMPLE 9 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

- Find the position function giving the height  $s$  as a function of the time  $t$ .
- When does the ball hit the ground?

#### Solution

- Let  $t = 0$  represent the initial time. The two given initial conditions can be written as follows.

$$s(0) = 80$$

Initial height is 80 feet.

$$s'(0) = 64$$

Initial velocity is 64 feet per second.

Using  $-32$  feet per second per second as the acceleration due to gravity, you can write

$$s''(t) = -32$$

$$s'(t) = \int s''(t) dt = \int -32 dt = -32t + C_1.$$

Using the initial velocity, you obtain  $s'(0) = 64 = -32(0) + C_1$ , which implies that  $C_1 = 64$ . Next, by integrating  $s'(t)$ , you obtain

$$s(t) = \int s'(t) dt = \int (-32t + 64) dt = -16t^2 + 64t + C_2.$$

Using the initial height, you obtain

$$s(0) = 80 = -16(0)^2 + 64(0) + C_2$$

which implies that  $C_2 = 80$ . So, the position function is


$$s(t) = -16t^2 + 64t + 80. \quad \text{See Figure 4.4.}$$

- Using the position function found in part (a), you can find the time at which the ball hits the ground by solving the equation  $s(t) = 0$ .

$$-16t^2 + 64t + 80 = 0$$

$$-16(t + 1)(t - 5) = 0$$

$$t = -1, 5$$

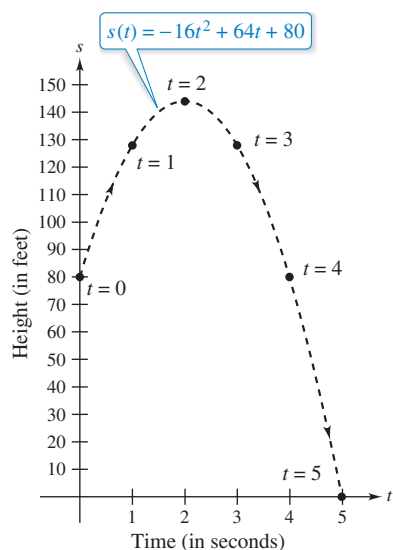
Because  $t$  must be positive, you can conclude that the ball hits the ground 5 seconds after it was thrown. 

In Example 9, note that the position function has the form

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$

where  $g = -32$ ,  $v_0$  is the initial velocity, and  $s_0$  is the initial height, as presented in Section 2.2.

Example 9 shows how to use calculus to analyze vertical motion problems in which the acceleration is determined by a gravitational force. You can use a similar strategy to analyze other linear motion problems (vertical or horizontal) in which the acceleration (or deceleration) is the result of some other force, as you will see in Exercises 61–68.



Height of a ball at time  $t$

Figure 4.4

## 4.1 Exercises

See [CalcChat.com](http://CalcChat.com) for tutorial help and worked-out solutions to odd-numbered exercises.

**Integration and Differentiation** In Exercises 1 and 2, verify the statement by showing that the derivative of the right side equals the integrand of the left side.

- $\int \left(-\frac{6}{x^4}\right) dx = \frac{2}{x^3} + C$
- $\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2x} + C$

**Solving a Differential Equation** In Exercises 3–6, find the general solution of the differential equation and check the result by differentiation.

- $\frac{dy}{dt} = 9t^2$
- $\frac{dy}{dt} = 5$
- $\frac{dy}{dx} = x^{3/2}$
- $\frac{dy}{dx} = 2x^{-3}$

**Rewriting Before Integrating** In Exercises 7–10, complete the table to find the indefinite integral.

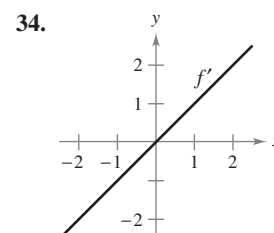
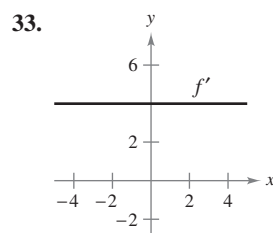
Original Integral	Rewrite	Integrate	Simplify
7. $\int \sqrt[3]{x} dx$			
8. $\int \frac{1}{4x^2} dx$			
9. $\int \frac{1}{x\sqrt{x}} dx$			
10. $\int \frac{1}{(3x)^2} dx$			

**Finding an Indefinite Integral** In Exercises 11–32, find the indefinite integral and check the result by differentiation.

- $\int (x + 7) dx$
- $\int (13 - x) dx$
- $\int (x^5 + 1) dx$
- $\int (8x^3 - 9x^2 + 4) dx$
- $\int (x^{3/2} + 2x + 1) dx$
- $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$
- $\int \sqrt[3]{x^2} dx$
- $\int (\sqrt[4]{x^3} + 1) dx$
- $\int \frac{1}{x^5} dx$
- $\int \frac{3}{x^7} dx$
- $\int \frac{x + 6}{\sqrt{x}} dx$
- $\int \frac{x^4 - 3x^2 + 5}{x^4} dx$
- $\int (x + 1)(3x - 2) dx$
- $\int (4t^2 + 3)^2 dt$
- $\int (5 \cos x + 4 \sin x) dx$
- $\int (t^2 - \cos t) dt$
- $\int (1 - \csc t \cot t) dt$
- $\int (\theta^2 + \sec^2 \theta) d\theta$
- $\int (\sec^2 \theta - \sin \theta) d\theta$
- $\int \sec y (\tan y - \sec y) dy$

- $\int (\tan^2 y + 1) dy$
- $\int (4x - \csc^2 x) dx$

**Sketching a Graph** In Exercises 33 and 34, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.) To print an enlarged copy of the graph, go to [MathGraphs.com](http://MathGraphs.com).



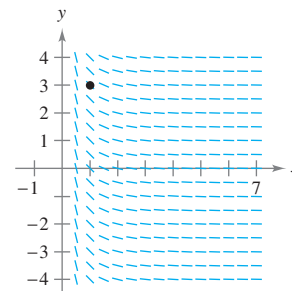
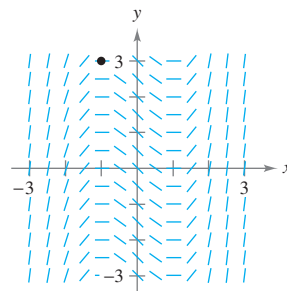
**Finding a Particular Solution** In Exercises 35–42, find the particular solution that satisfies the differential equation and the initial condition.

- $f'(x) = 6x, f(0) = 8$
- $g'(x) = 4x^2, g(-1) = 3$
- $h'(t) = 8t^3 + 5, h(1) = -4$
- $f'(s) = 10s - 12s^3, f(3) = 2$
- $f''(x) = 2, f'(2) = 5, f(2) = 10$
- $f''(x) = x^2, f'(0) = 8, f(0) = 4$
- $f''(x) = x^{-3/2}, f'(4) = 2, f(0) = 0$
- $f''(x) = \sin x, f'(0) = 1, f(0) = 6$



**Slope Field** In Exercises 43 and 44, a differential equation, a point, and a slope field are given. A *slope field* (or *direction field*) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (To print an enlarged copy of the graph, go to [MathGraphs.com](http://MathGraphs.com).) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

- $\frac{dy}{dx} = x^2 - 1, (-1, 3)$
- $\frac{dy}{dx} = -\frac{1}{x^2}, x > 0, (1, 3)$





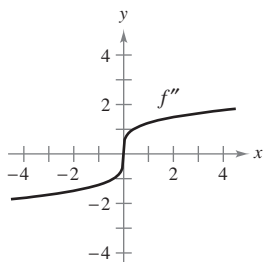


**Slope Field** In Exercises 45 and 46, (a) use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.

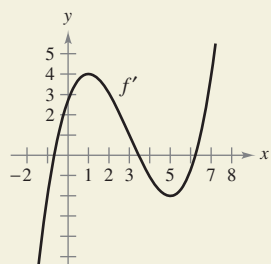
45.  $\frac{dy}{dx} = 2x$ ,  $(-2, -2)$       46.  $\frac{dy}{dx} = 2\sqrt{x}$ ,  $(4, 12)$

### WRITING ABOUT CONCEPTS

47. **Antiderivatives and Indefinite Integrals** What is the difference, if any, between finding the antiderivative of  $f(x)$  and evaluating the integral  $\int f(x) dx$ ?
48. **Comparing Functions** Consider  $f(x) = \tan^2 x$  and  $g(x) = \sec^2 x$ . What do you notice about the derivatives of  $f(x)$  and  $g(x)$ ? What can you conclude about the relationship between  $f(x)$  and  $g(x)$ ?
49. **Sketching Graphs** The graphs of  $f$  and  $f'$  each pass through the origin. Use the graph of  $f''$  shown in the figure to sketch the graphs of  $f$  and  $f'$ . To print an enlarged copy of the graph, go to [MathGraphs.com](http://MathGraphs.com).



50. **HOW DO YOU SEE IT?** Use the graph of  $f'$  shown in the figure to answer the following.



- Approximate the slope of  $f$  at  $x = 4$ . Explain.
- Is it possible that  $f(2) = -1$ ? Explain.
- Is  $f(5) - f(4) > 0$ ? Explain.
- Approximate the value of  $x$  where  $f$  is maximum. Explain.
- Approximate any open intervals in which the graph of  $f$  is concave upward and any open intervals in which it is concave downward. Approximate the  $x$ -coordinates of any points of inflection.

51. **Tree Growth** An evergreen nursery usually sells a certain type of shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by  $dh/dt = 1.5t + 5$ , where  $t$  is the time in years and  $h$  is the height in centimeters. The seedlings are 12 centimeters tall when planted ( $t = 0$ ).

- Find the height after  $t$  years.
- How tall are the shrubs when they are sold?

52. **Population Growth** The rate of growth  $dP/dt$  of a population of bacteria is proportional to the square root of  $t$ , where  $P$  is the population size and  $t$  is the time in days ( $0 \leq t \leq 10$ ). That is,

$$\frac{dP}{dt} = k\sqrt{t}.$$

The initial size of the population is 500. After 1 day the population has grown to 600. Estimate the population after 7 days.

**Vertical Motion** In Exercises 53–55, use  $a(t) = -32$  feet per second per second as the acceleration due to gravity. (Neglect air resistance.)

- A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?
- With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?
- A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.
  - How many seconds after its release will the bag strike the ground?
  - At what velocity will it hit the ground?

**Vertical Motion** In Exercises 56–58, use  $a(t) = -9.8$  meters per second per second as the acceleration due to gravity. (Neglect air resistance.)

- A baseball is thrown upward from a height of 2 meters with an initial velocity of 10 meters per second. Determine its maximum height.
- With what initial velocity must an object be thrown upward (from a height of 2 meters) to reach a maximum height of 200 meters?

- • • 58. **Grand Canyon** • • • • •
- The Grand Canyon is 1800 meters deep at its deepest point. A rock is dropped from the rim above this point. Write the height of the rock as a function of the time  $t$  in seconds. How long will it take the rock to hit the canyon floor?



- 59. Lunar Gravity** On the moon, the acceleration due to gravity is  $-1.6$  meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

- 60. Escape Velocity** The minimum velocity required for an object to escape Earth's gravitational pull is obtained from the solution of the equation

$$\int v \, dv = -GM \int \frac{1}{y^2} \, dy$$

where  $v$  is the velocity of the object projected from Earth,  $y$  is the distance from the center of Earth,  $G$  is the gravitational constant, and  $M$  is the mass of Earth. Show that  $v$  and  $y$  are related by the equation

$$v^2 = v_0^2 + 2GM \left( \frac{1}{y} - \frac{1}{R} \right)$$

where  $v_0$  is the initial velocity of the object and  $R$  is the radius of Earth.

**Rectilinear Motion** In Exercises 61–64, consider a particle moving along the  $x$ -axis where  $x(t)$  is the position of the particle at time  $t$ ,  $x'(t)$  is its velocity, and  $x''(t)$  is its acceleration.

- 61.**  $x(t) = t^3 - 6t^2 + 9t - 2$ ,  $0 \leq t \leq 5$
- Find the velocity and acceleration of the particle.
  - Find the open  $t$ -intervals on which the particle is moving to the right.
  - Find the velocity of the particle when the acceleration is 0.
- 62.** Repeat Exercise 61 for the position function
- $$x(t) = (t - 1)(t - 3)^2, \quad 0 \leq t \leq 5.$$
- 63.** A particle moves along the  $x$ -axis at a velocity of  $v(t) = 1/\sqrt{t}$ ,  $t > 0$ . At time  $t = 1$ , its position is  $x = 4$ . Find the acceleration and position functions for the particle.
- 64.** A particle, initially at rest, moves along the  $x$ -axis such that its acceleration at time  $t > 0$  is given by  $a(t) = \cos t$ . At the time  $t = 0$ , its position is  $x = 3$ .
- Find the velocity and position functions for the particle.
  - Find the values of  $t$  for which the particle is at rest.
- 65. Acceleration** The maker of an automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assume the acceleration is constant.
- Find the acceleration in meters per second per second.
  - Find the distance the car travels during the 13 seconds.
- 66. Deceleration** A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.
- How far has the car moved when its speed has been reduced to 30 miles per hour?
  - How far has the car moved when its speed has been reduced to 15 miles per hour?
  - Draw the real number line from 0 to 132. Plot the points found in parts (a) and (b). What can you conclude?

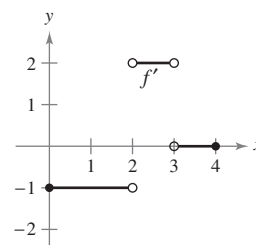
- 67. Acceleration** At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.

- How far beyond its starting point will the car pass the truck?
- How fast will the car be traveling when it passes the truck?

- 68. Acceleration** Assume that a fully loaded plane starting from rest has a constant acceleration while moving down a runway. The plane requires 0.7 mile of runway and a speed of 160 miles per hour in order to lift off. What is the plane's acceleration?

**True or False?** In Exercises 69–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- 69.** The antiderivative of  $f(x)$  is unique.
- 70.** Each antiderivative of an  $n$ th-degree polynomial function is an  $(n + 1)$ th-degree polynomial function.
- 71.** If  $p(x)$  is a polynomial function, then  $p$  has exactly one antiderivative whose graph contains the origin.
- 72.** If  $F(x)$  and  $G(x)$  are antiderivatives of  $f(x)$ , then
- $$F(x) = G(x) + C.$$
- 73.** If  $f'(x) = g(x)$ , then  $\int g(x) \, dx = f(x) + C$ .
- 74.**  $\int f(x)g(x) \, dx = \int f(x) \, dx \int g(x) \, dx$
- 75. Horizontal Tangent** Find a function  $f$  such that the graph of  $f$  has a horizontal tangent at  $(2, 0)$  and  $f''(x) = 2x$ .
- 76. Finding a Function** The graph of  $f'$  is shown. Find and sketch the graph of  $f$  given that  $f$  is continuous and  $f(0) = 1$ .



- 77. Proof** Let  $s(x)$  and  $c(x)$  be two functions satisfying  $s'(x) = c(x)$  and  $c'(x) = -s(x)$  for all  $x$ . If  $s(0) = 0$  and  $c(0) = 1$ , prove that  $[s(x)]^2 + [c(x)]^2 = 1$ .

### PUTNAM EXAM CHALLENGE

- 78.** Suppose  $f$  and  $g$  are non-constant, differentiable, real-valued functions defined on  $(-\infty, \infty)$ . Furthermore, suppose that for each pair of real numbers  $x$  and  $y$ ,

$$f(x + y) = f(x)f(y) - g(x)g(y) \quad \text{and} \\ g(x + y) = f(x)g(y) + g(x)f(y).$$

If  $f'(0) = 0$ , prove that  $(f(x))^2 + (g(x))^2 = 1$  for all  $x$ .

This problem was composed by the Committee on the Putnam Prize Competition.  
© The Mathematical Association of America. All rights reserved.